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Wittgenstein's Annotations to Hardy's Course of Pure Mathematics: An Investigation of Wittgenstein's Non-Extensionalist Understanding of the Real Numbers by Juliet Floyd and Felix Mühlhölzer

Kim Solin

Floyd, Juliet and Felix Mühlhölzer: *Wittgenstein's Annotations to Hardy's Course of Pure Mathematics: An Investigation of Wittgenstein's Non-Extensionalist Understanding of the Real Numbers*. Nordic Wittgenstein Studies. Cham: Springer Verlag 2020, 322 pp.

Around 1942–1943 Wittgenstein read G.H. Hardy's book *A Course of Pure Mathematics* (8th edition, published 1941) and made annotations in the margin. The book was a standard university text at the time, on introductory but still fairly advanced topics, written by an established mathematician. So here we find Wittgenstein commenting directly on more advanced mathematics presented by a real mathematician. Wittgenstein's copy of Hardy's book is lost, but there are photocopies of the pages containing his annotations, of which images are included in the book together with transcriptions and translations. Some of the annotations were elaborated in Wittgenstein's manuscripts and some of these elaborations were, in turn, included in the *Remarks on the Foundations of Mathematics*. The editors of the *Remarks* had considered including more of them in the revised edition of 1974, with explicit references to Hardy's book, but ultimately decided not to (cf. Solin *forth.* 2022). The authors of *Wittgenstein's Annotations* skilfully expound the annotations, the related manuscripts and the relevant parts of the *Remarks*, and focus on what they call Wittgenstein's non-extensionalist perspective.

By Wittgenstein's non-extensionalist perspective, the authors mean the following:

> With the term "extension" Wittgenstein has two things in mind. First, he will strictly distinguish between sequences of numbers that the extensionalist considers to be, in Cantor's sense, "finished" [fertig]

> entities or sets – these are the "extensions" – from the techniques or rules by means of which such entities may be produced, assessed, or accessed. If there are such techniques, the extensionalist's interest is ultimately only in their results, the produced sequences, and not the possible processes or conceptual motifs or definitions leading to them. […] For the non-extensionalist, on the other hand, it is the processes and structured conceptual motifs, the grammar or logic of the notions, we should be concerned with. […] The second context in which Wittgenstein speaks of "extensions" is the context of sets, paradigmatically sets of numbers: N, Q, R, and subsets of them. Considered extensionally, the laws or rules or techniques through which we may approach them are taken as irrelevant, as their identity is only determined by the elements of which they consist. From Wittgenstein's non-extensionalist view, however, it is precisely these laws, rules or techniques we should take as primary. (pp. 31–32)

Floyd and Mühlhölzer show, I believe convincingly, that throughout Wittgenstein's writings, from the *Tractatus* to at least 1947, he is concerned with the relation between the extensionalist and the non-extensionalist perspective.

Contrary to what one might expect, the book is not only about the annotations to Hardy's *Course* and about the non-extensionalist perspective. In expounding the annotations, the authors give nothing less than a wide-ranging introduction to their understanding of Wittgenstein's philosophy of mathematics and, perhaps, of their view on his philosophy as a whole. They give a historical description of the development of Wittgenstein's thoughts on the subject, make good use of the today openly available *Nachlass* manuscripts and typescripts, critically discuss the choices made by the editors of the *Remarks*, and scrutinize parts of the original translation. The discussion is mathematically detailed and accurate, and relates to a selection of the secondary literature without drowning in it. This makes the book a very rich work indeed.

More specifically, the book has the following structure. The authors begin in Part I by giving a general overview, placing the annotations in historical context and explaining what they mean by the non-extensional point of view. In the second part, they analyse Wittgenstein's annotations in three jointly written chapters on irrational numbers and Dedekind cuts, the continuum of real numbers, and on functions and limits, respectively. In passing, the authors also devote a chapter to the law of excluded middle. When analysing the annotations, the authors discuss, in relation to the non-extensionalist point of view, among other things the difference between illustration and application, the difference between proper application and imagined ("phantasmagorical") application, and the important difference between prose and calculus. The authors point out that although Wittgenstein for example "toys with designing a totally new sort of mathematics dealing with continuity" (p. 63), he is not actually doing mathematics, but instead presents an advanced and stringent object of comparison to create a contrast to what mathematicians actually do, in order to make the actual practice clearer.

The book concludes in Part III with two separately written chapters on Cantor's diagonal proof, also showing, as the authors say in the Preface, that they have "somewhat different attitudes towards the ultimate philosophical position" (p. vii). Mühlhölzer's chapter is a detailed, instructive and knowledgeable exegesis of Part II of the revised edition of the *Remarks*. He identifies what he sees as the central issue of these remarks of Wittgenstein's, namely, "the disparity between the simplicity of Cantor's idea, on the one hand, and the supposedly deep and amazing mathematical results gained by it, on the other" (p. 191). Importantly, Mühlhölzer argues that reading certain remarks from extensionalist perspective can occlude what Wittgenstein wants to show us, whereas reading them from the non-extensionalist perspective lets us see the "richness of Wittgenstein's non-extensionalist perspective" (p. 146). The chapter also contains a critical discussion of the editors' composition of Part II, including the aforementioned scrutiny of Anscombe's translation. The discussion involves MS 162, which the editors had considered including in the revised edition of the *Remarks*, but in the end left out (cf. Solin *forth.* 2022).

Floyd's chapter is a discussion of the generality of Cantor's method in the sense of a proof technique that can varied and deployed in different contexts, 'technique' being a classical example of a family-resemblance concept. Interpreting Wittgenstein, Floyd distinguishes between aspects and techniques, in short, "aspects are *discovered* or noticed or revealed, whereas techniques are *invented*" (p. 208). This is the non-extensionalist understanding of the generality of Cantor's method, in contrast to the extensionalist conception, which would involve "a sharply expressed quantificational understanding of how far we may generalize over 'all' real numbers, in terms of scope". The whole idea with the extensional perspective, Floyd writes (p. 205), is to eliminate any reference to technique, in contrast to the non-extensionalist perspective, which aims to describe these different techniques. Nevertheless, according to Floyd (and perhaps *contra* Mühlhölzer, cf. above), "Wittgenstein does not wish to reject or defeat the more abstract extensional perspective in favour of the non-extensional perspective: he does not see modern mathematics as involving us in a 'fight'. Rather he is interested in comparing and investigating the different approaches to the real numbers" (p. 218). Wittgenstein's remarks thus become, in the sense I described above, detailed objects of comparison, but the extensionalist perspective does not blind us, it is just another way of looking at things.

Central to Floyd's chapter is a discussion of Alan Turing's seminal 1936 paper, which introduces what are today called Turing machines and in which Turing employs a kind of diagonal argument. Turing, according to Floyd, fruitfully works with both the extensionalist and non-extensionalist perspective (pp. 227–228). Wittgenstein's interaction with Turing, especially in the summer of 1937, is given an important role. Moreover, Floyd suggests that Turing was acquainted with and drew ideas from *The Blue and the Brown Books* for his ground-breaking paper (p. 243). Using MS 162, Floyd also shows that Wittgenstein commented on more advanced mathematics than that which he presented in his lectures (p. 249). As late as 1947, Floyd argues, Wittgenstein even devised his own diagonal argument by expressing Turing's diagonal argument using language-game terminology (p. 257). One of the merits of this chapter – and of the whole book – is that it shows just how *fingerfertig* Wittgenstein actually was with the technical details of theories he was discussing. Nevertheless, perhaps the significance of Turing's work is another point where the authors differ slightly, since if I understand them correctly, Floyd is of the view that Turing essentially resolved Hilbert's *Entscheidungsproblem* (p. 238), whereas Mühlhölzer emphasises the inherent vagueness of it all (pp. 168–169). However that may be, Floyd's chapter is a crystal contribution to research on Turing's work and on its relation to Wittgenstein.

There is one aspect of the book that I find problematic and that I am not quite sure what to say about. It concerns the following joint statement by the authors, here condensed by me:

> One must admit at the outset that Wittgenstein's criticisms of CPM [Hardy's *Course of Pure Mathematics*, ann. KS] are, unfortunately, pervaded by a subliminal resentment that is characteristic of many of his investigations of mathematics. […] He is suspicious of the characteristically abstract traits of mathematics as they arose through the work of Cantor and Dedekind in the nineteenth century. […] Most of the time Wittgenstein's suspicions are kept in check, working only subliminally, and evincing themselves in clipped and brief phrases. But sometimes they break out and bubble up to the surface. […] When reading Wittgenstein it is important not to be dragged into sharing and promulgating this resentment. We do not think, however, that it is all that difficult to guard against it and set it to one side. The thoughts that remain are, we think, mostly perceptive

> and worth taking seriously and in any case should not be set aside on the grounds of such *ressentiment*, especially given the working, private nature of the manuscripts that were never intended for publication as they stand. (pp. 22–23)

This is a reconciling gesture towards mathematicians who have been offended by Wittgenstein's philosophy of mathematics. As is well known, the reception of the *Remarks* when they were first published was less than enthusiastic among mathematicians and logicians. It is laudable that the authors try to do something about this. Going about it this way is unfortunate, though, since what the authors call "subliminal resentment", but which could simply be called critique, is nevertheless there to be found in Wittgenstein's manuscripts to an extent that it cannot be ignored. The authors say that remarks of this kind were not intended for publication as they stand. But how are we to determine which remarks were intended for publication and which not? And even if we knew how to, does the question of suitability for academic publication really matter for understanding Wittgenstein's thought as an integral whole? A discussion focused on remarks of the critical kind (for example, RFM, II, § 23, written in May 1938; or RFM, VII, § 19, written in June–September 1941, see also the underlying MS 124 for an alternative formulation) would have made for a different kind of book. To do justice to Wittgenstein's thinking as a whole, one would need to incorporate and discuss both kinds of remarks, otherwise the discussion becomes one-sided. But one gets the impression that the authors of *Wittgenstein's Annotations* do not really know what to make of the critical kind of remarks and therefore simply ignores them, sets them to one side as irrelevant to the topic under discussion. In this regard, the book is disappointing. I do not think that simply ignoring the critical remarks and making them seem like temporary whims, or like Wittgenstein was having a bad day and was tired and cranky, as the authors suggest in the Preface (p. x), does justice to Wittgenstein's writings on mathematics taken as a whole. It is also symptomatic that Rush Rhees, who had written on continuity and discussed it with Wittgenstein at length in August 1938 (a fact not even mentioned by the authors; see Rhees 1970), who was chosen as a literary executor, and who often emphasized the critical side of Wittgenstein's thought, is presented as little more than a clumsy editor. There might be a tension here, though, since Mühlhölzer in his own chapter elaborates and defends one of Wittgenstein's perhaps more offensive statements, "I believe, and hope, that a future generation will laugh at this hocus pocus" (RFM, II, §22), and laments that it had been left out in the first edition "as if it were something to be ashamed of – for Wittgenstein himself and for those who want to take his philosophy seriously" (p. 177). Could one not think that other remarks of that tone at closer inspection are more reasonable and rewarding than they might first appear to be?

For the one willing or even wishing to set the critical remarks aside, or perhaps honestly is simply not able to engage with them, the book remains – on the level it is written (cf. RFM, VI, § 12, written sometime 1941–44) – highly conscientious and detailed work. For the scholar who wants to take this further and make more broad sense of Wittgenstein's philosophy of mathematics – and try to give an answer to *why* it was so central to Wittgenstein to think about mathematics the way he did – this book forms a solid foundation in that it has instructively presented and sorted out many technical details and misconceptions, making good use of the *Nachlass*.

Uppsala University, Sweden
kim.solin(at)uu.se

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